

HIGHER ITERATED HILBERT COEFFICIENTS OF THE GRADED COMPONENTS OF BIGRADED MODULES

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ABSTRACT. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over the field K , and let $I \subset S$ be a graded ideal. It is shown that the higher iterated Hilbert coefficients of the graded S -modules $\text{Tor}_i^S(M, I^k)$ and $\text{Ext}_S^i(M, I^k)$ are polynomial functions in k , and an upper bound for their degree is given. These results are derived by considering suitable bigraded modules.

INTRODUCTION

The present paper is motivated by Kodiyalam's work [6], the papers by Theodorescu [11], by Katz and Theodorescu [8], [9] and the paper [3]. In these papers it was shown that for finitely generated R -modules M and N over a Noetherian (local) ring R , and for an ideal $I \subset R$ such that the length of $\text{Tor}_i^R(M, N/I^k N)$ is finite for all k , it follows that the length of $\text{Tor}_i^R(M, N/I^k N)$ is eventually a polynomial function in k . In these papers bounds are given for the degree of these polynomials. In some cases also the leading coefficient is determined. Similar results have been proved for the Ext-modules.

In this paper we consider a related problem. Here $I \subset S$ is graded ideal and S is the polynomial ring. It is shown in Corollary 3.2 that for any finitely generated graded S -module M , the modules $\text{Tor}_i^S(M, I^k)$ are finitely graded S -modules which for $k \gg 0$ have constant Krull dimension, and furthermore in Corollary 3.5 it is shown that the higher iterated Hilbert coefficients (which appear as the coefficients of the higher iterated Hilbert polynomials) are all polynomials functions. A related result has been shown in [4] for the case $M/I^k M$ and in [5] for the case $\text{Tor}_i^S(S/\mathfrak{m}, I^k)$, where \mathfrak{m} denotes the graded maximal ideal of S .

Observe that knowing all higher iterated Hilbert coefficients of a graded module is equivalent to knowing its h -vector, and hence the Hilbert series of the module. This is the reason why we are not only interested in the ordinary Hilbert coefficients, but in all higher iterated Hilbert coefficients.

For the proof we use a technique which was first introduced by Kodiyalam [7]. For this purpose we consider the bigraded K -algebra $A = K[x_1, \dots, x_n, y_1, \dots, y_m]$ with $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$ for all i and j , and a finitely generated bigraded A -module M . A typical example of such an A -module is the Rees algebra of a graded ideal $I \subset S$ with $I = (f_1, \dots, f_m)$ and $\deg f_j = p_j$ for all j . For each k , the S -module $M_k = \bigoplus_i M_{(i,k)}$ is a finitely generated graded S -module. A graded free S -resolution of M_k can be obtained by the graded components of the bigraded free A -module resolution of M . These resolutions are then used to compute the higher iterated Hilbert polynomials of the graded S -modules M .

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The first (and important step) is to show that the higher iterated Hilbert coefficients of the components $A(-a, -b)_k$ of the bi-shifted free A -module $A(-a, -b)$ are polynomial functions in k for $k \gg 0$, see Proposition 1.5. This result and the bigraded resolution \mathbb{F} of the bigraded A -module M is then used in the next section to prove the same result for M . There, by using a graded version of the Noether Normalization Theorem, one obtains in Theorem 2.2 and Theorem 2.5 upper bounds for the degree of these polynomials. The better bound for the degree of the polynomial function representing the Hilbert coefficient $e_j^i(M_k)$ is achieved when all p_t are the same and it is given by $\deg e_j^i(M_k) \leq \dim M/\mathfrak{m}M + j - 1$. These results are then applied in Section 3 to show that for any finitely generated graded S -module M , and any finitely generated bigraded module N , the higher iterated Hilbert coefficients of the graded S -modules $\mathrm{Tor}_i^S(M, N_k)$ and $\mathrm{Ext}_S^i(M, N_k)$ are polynomial functions in k for $k \gg 0$.

1. THE GRADED COMPONENTS OF A BIGRADED MODULE AND THEIR HIGHER ITERATED HILBERT COEFFICIENTS

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables with the standard grading. Let $A = K[x_1, \dots, x_n, y_1, \dots, y_m]$ with bigrading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$, for some integers $p_j \geq 0$.

For a finitely generated bigraded A -module $M = \bigoplus_{i,j \in \mathbb{Z}} M_{(i,j)}$, we define M_k to be the graded S -module $\bigoplus_{i \in \mathbb{Z}} M_{(i,k)}$.

These definitions are motivated by the following important class of examples: Let $I \subset S$ be a graded ideal generated by the homogeneous polynomials f_1, \dots, f_m with $\deg f_j = p_j$. Then the Rees ring $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k$ is bigraded A -module with $\mathcal{R}(I)_k = I^k$ for all k .

For $a, b \in \mathbb{Z}$, the twisted module A -module $M(-a, -b)$ is defined to be the bigraded A -module with components $M(-a, -b)_{(i,j)} = M(i - a, j - b)$.

In this section we want to compute the Hilbert coefficients of the S -module $A(-a, -b)_k$ as a function of k .

Note that

$$(1) \quad A(-a, -b)_k = (A_{k-b})(-a) \cong \bigoplus_{\beta_1 + \dots + \beta_m = k-b} S(-(p_1\beta_1 + \dots + p_m\beta_m) - a)y_1^{\beta_1} \dots y_m^{\beta_m}.$$

Hence, in a first step, we have to determine the Hilbert coefficients of $S(-c)$ for some $c \in \mathbb{Z}$.

Recall that for a finite graded S -module M and all $k \gg 0$, the numerical function $H(M, k) = \dim_K M_k$ is called the Hilbert function of M . For $i \in \mathbb{N}$, the higher iterated Hilbert functions $H_i(M, k)$ are defined recursively as follows:

$$H_0(M, k) = H(M, k), \quad \text{and} \quad H_i(M, k) = \sum_{j \leq k} H_{i-1}(M, j).$$

By Hilbert it is known that $H_i(M, k)$ is of polynomial type of degree $d + i - 1$, where d is the Krull dimension of M . In other words, there exists a polynomial $P_M^i(x) \in \mathbb{Q}[x]$ of degree $d + i - 1$ such that $H_i(M, k) = P_M^i(k)$ for all $k \gg 0$. This unique polynomial is called the *ith Hilbert polynomial* of M . It can be written in the form

$$P_M^i(x) = \sum_{j=0}^{d+i-1} (-1)^j e_j^i(M) \binom{x + d + i - j - 1}{d + i - j - 1}$$

with integer coefficients $e_j^i(M)$, called the *higher iterated Hilbert coefficients* of M , where by definition

$$\binom{i}{j} = \frac{i(i-1)\cdots(i-j+1)}{j(j-1)\cdots 2\cdot 1} \quad \text{if } j > 0 \quad \text{and} \quad \binom{i}{0} = 1.$$

In the important special case when $M = S$ we have

$$P_S^i(x) = \binom{x+n+i-1}{n+i-1}.$$

More generally, if $c \in \mathbb{Z}$, then

$$(2) \quad P_{S(-c)}^i(x) = \binom{x-c+n+i-1}{n+i-1}$$

In particular, $\deg P_{S(-c)}^i(x) = n+i-1$.

In order to compute the higher iterated Hilbert coefficients of M , we define the *difference operator* Δ on the set of polynomial functions by setting $(\Delta P)(a) = P(a) - P(a-1)$ for all $a \in \mathbb{Z}$. The d times iterated Δ operator will be denoted by Δ^d . We further set $\Delta^0 P = P$.

For our further considerations we shall need the following easy lemma whose proof we omit.

Lemma 1.1. *Let $P(x) = \sum_{i=0}^n (-1)^i f_i \binom{x+n-i}{n-i}$. Then*

$$(\Delta^j P)(-1) = (-1)^{n-j} f_{n-j}, \quad \text{for } j = 0, 1, \dots, n.$$

Applying this formula to the higher iterated Hilbert polynomials of M we obtain

$$(3) \quad e_j^i(M) = (-1)^j \Delta^{d+i-j-1} P_M^i(-1) \quad \text{for } j = 0, \dots, d+i-1,$$

where $d = \dim M$.

Since $\Delta P_M^i = P_M^{i-1}$ for all $i \geq 1$, formula (3) yields

Corollary 1.2. $e_j^i(M) = e_j^{i-1}(M)$ for $j = 0, \dots, d+i-2$, where $d = \dim M$.

Having in mind Corollary 1.2, we set $e_j(M) = e_j^0(M)$ for $j = 0, \dots, d-1$, and $e_j(M) = e_j^{j-(d-1)}(M)$ for $j \geq d$. Then for all i it follows that $e_j^i(M) = e_j(M)$ for $j = 0, \dots, i+d-1$. Therefore,

$$P_M^i(x) = \sum_{j=0}^{d+i-1} (-1)^j e_j(M) \binom{x+d+i-j-1}{d+i-j-1}.$$

Let M be a finitely generated graded S -module of dimension d , generated in non-negative degrees, and let $H_M(t)$ be the Hilbert series of M . Then there exists a unique polynomial $Q_M(t) \in \mathbb{Q}[t]$ such that $H_M(t) = Q_M(t)/(1-t)^d$. Let $Q_M(t) = \sum_{i=0}^s h_i t^i$ with $h_s \neq 0$. The coefficient vector (h_0, \dots, h_s) of $Q_M(t)$ is called the h -vector of M . The following relation between the iterated Hilbert coefficients and the h -vector of M is well known.

- (i) $e_j = \sum_{i=j}^s \binom{i}{j} h_i$ for $j = 0, \dots, s$ and $e_j = 0$ for $j > s$.
- (ii) $h_i = \sum_{j=i}^s (-1)^{j-i} \binom{j}{i} e_j$ for $i = 1, \dots, s$.

These relations show that the set of higher iterated Hilbert coefficients determine the Hilbert series of M completely.

Proposition 1.3. *Let $c \in \mathbb{Z}$. Then the higher iterated Hilbert coefficients of $S(-c)$ are*

$$e_j^i(S(-c)) = \binom{c}{j} \quad \text{for all } i \geq 0 \text{ and all } j \text{ with } 0 \leq j \leq n + i - 1.$$

In particular, $e_j^i(S(-c)) = 0$ if and only if $0 \leq c < j \leq n + i - 1$.

Proof. We have $\Delta^j P_{S(-c)}^i(x) = \binom{x-c+n+i-j-1}{n+i-j-1}$, and hence by formula (3) we get

$$e_j^i(S(-c)) = (-1)^j \binom{j-c-1}{j} = \binom{c}{j}.$$

□

Now by using Proposition 1.3 we can give an upper bound for the higher iterated Hilbert coefficients of $A(-a, -b)_k$. Before that we need an elementary lemma.

Lemma 1.4. *Let $P(x) \in Q[x]$ be a polynomial of degree d . Then $F(k) = \sum_{j=0}^k P(j)$ is a polynomial in k of degree $d + 1$.*

Proof. Let $P(x) = \sum_{i=0}^d a_i x^i$. Then $F(k) = \sum_{j=0}^k \sum_{i=0}^d a_i j^i = \sum_{i=0}^d a_i (\sum_{j=0}^k j^i)$. It is well-known that $\sum_{j=0}^k j^i$ is a polynomial in k of degree $i + 1$. So $F(k)$ is a polynomial in k of degree $d + 1$. □

In proof of the next proposition we use Lemma 1.4 and also the fact that for all a, c, j we have

$$\binom{c+a}{j} = \sum_{i=0}^a \binom{a}{i} \binom{c}{j-i}.$$

Proposition 1.5. *For $k \gg 0$, the higher iterated Hilbert coefficients $e_j^i(A(-a, -b)_k)$ are polynomial functions of degree $m + j - 1$ with*

$$e_j^i(A(-a, -b)_k) \leq \binom{k-b+m-1}{m-1} \binom{p_m(k-b)+a}{j}.$$

Equality holds, if and only if $p_1 = p_2 = \dots = p_m$ for all j .

Proof. Without restriction we may assume that $p_1 \leq p_2 \leq \dots \leq p_m$.

For $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{Z}_{\geq 0}^m$ and $p = (p_1, \dots, p_m)$ we set $|\beta| = \sum_{i=1}^m \beta_i$ and $p\beta = \sum_{i=1}^m p_i \beta_i$. Furthermore, let $C(k-b) = \{\beta : |\beta| = k-b\}$.

By (1), the S -module $A(-a, -b)_k$ is the direct sum of the shifted free S -modules $S(-p\beta - a)$ with $\beta \in C(k-b)$.

Therefore,

$$P_{A(-a, -b)_k}^i(x) = \sum_{\beta \in C(k-b)} P_{S(-p\beta - a)}^i(x).$$

Since $\deg P_{S(-p\beta - a)}^i(x) = n + i - 1$ for all $\beta \in C(k-b)$, by Proposition 1.3 we get

$$\begin{aligned} (4) \quad e_j^i(A(-a, -b)_k) &= \sum_{\beta \in C(k-b)} e_j^i(S(-p\beta - a)) \\ &= \sum_{\beta \in C(k-b)} \binom{p\beta + a}{j}. \end{aligned}$$

We show by induction on m that $\sum_{\beta \in C(k-b)} \binom{p\beta+a}{j}$ is a polynomial function in k . In order to let the induction work we actually show more generally that $\sum_{\beta \in C(k-b)} \binom{p\beta+\ell(k)}{j}$ is polynomial in k where $\ell(k)$ is linear function of k .

If $m = 1$, then

$$\begin{aligned} \sum_{\beta_1=0}^{k-b} \binom{p_1\beta_1 + \ell(k)}{j} &= \sum_{\beta_1=0}^{k-b} \sum_{i=0}^j \binom{p_1\beta_1}{j-i} \binom{\ell(k)}{i} \\ &= \sum_{i=0}^j \binom{\ell(k)}{i} \sum_{\beta_1=0}^{k-b} \binom{p_1\beta_1}{j-i} \end{aligned}$$

is polynomial in k , because, by (1.4), $\sum_{\beta_1=0}^{k-b} \binom{p_1\beta_1}{j-i}$ is polynomial in k . Now assume that $m > 1$. We set

$$F(k-b) = \sum_{\beta \in C(b-k)} \binom{p\beta + \ell(k)}{j}.$$

Then

$$\begin{aligned} F(k-b) &= \sum_{\beta_1=0}^{k-b} \sum_{\beta' \in C'(b-k-\beta_1)} \binom{p'\beta' + \ell(k)}{j} \\ &= \sum_{\beta_1=0}^{k-b} F'(k-b-\beta_1) \end{aligned}$$

where $\beta' = (\beta_2, \dots, \beta_m)$, $p' = (p_2 - p_1, \dots, p_m - p_1)$, $\ell'(k) = \ell(k) + p_1(k-b)$, $C'(k-b-\beta_1) = \{\beta' \mid |\beta'| = k-b-\beta_1\}$ and $F'(k-b-\beta_1) = \sum_{\beta' \in C'(b-k-\beta_1)} \binom{p'\beta' + \ell'(k)}{j}$.

By our induction hypothesis, $F'(k-b-\beta_1)$ is polynomial in k . Therefore by (1.4), $\sum_{\beta_1=0}^{k-b} F'(k-b-\beta_1) = \sum_{i=0}^{k-b} F'(i)$ is polynomial in k .

These considerations together with show that $e_j^i(A(-a, -b)_k)$ is polynomial function in k . Since

$$\binom{k-b+m-1}{m-1} \binom{p_1(k-b)+a}{j} \leq e_j^i(A(-a, -b)_k) \leq \binom{k-b+m-1}{m-1} \binom{p_m(k-b)+a}{j},$$

and since these lower and upper bounds are polynomial functions of degree $m+j-1$ with non-negative leading coefficient, we conclude that the degree of the polynomial functions $e_j^i(A(-a, -b)_k)$ is $m+j-1$, as well. Furthermore, it follows that $e_j^i(A(-a, -b)_k) = \binom{k-b+m-1}{m-1} \binom{p_m(k-b)+a}{j}$ if all p_t are the same.

Conversely, since $\binom{p\beta+a}{j} \leq \binom{p_m(k-b)+a}{j}$ for all summands $\binom{p\beta+a}{j}$ of $e_j^i(A(-a, -b)_k)$ it follows that $e_j^i(A(-a, -b)_k) = \binom{k-b+m-1}{m-1} \binom{(k-b)p_m+a}{j}$ if and only if $\binom{p\beta+a}{j} = \binom{p_m(k-b)+a}{j}$ for all $\beta \in (k-b)$. In particular, if $\beta = (k-b, 0, \dots, 0)$, then $\binom{p\beta+a}{j} = \binom{p_1(k-b)+a}{j} = \binom{p_m(k-b)+a}{j}$. It follows that $p_1 = p_m$, and hence $p_i = p_m$ for all i . \square

2. THE HIGHER ITERATED HILBERT COEFFICIENTS OF THE GRADED COMPONENTS OF A BIGRADED A -MODULE

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables with the standard grading, and let as before $A = K[x_1, \dots, x_n, y_1, \dots, y_m]$ be the polynomial ring with bigrading defined by $\deg x_i = (1, 0)$ and $\deg y_j = (p_j, 1)$, for some integers $p_j \geq 0$.

Let M be a finitely generated bigraded A -module. As before we set $M_k = \bigoplus_i M_{(i,k)}$. Then each M_k is a finitely generated graded S -module. In this section we want to study the higher iterated Hilbert coefficients $e_j^i(M_k)$. We set $\mathfrak{m} = (x_1, \dots, x_n)$ and $\mathfrak{n} = (y_1, \dots, y_m)$. Then $A/\mathfrak{n} = S$ and $A/\mathfrak{m} = S'$ where is the polynomial ring $K[y_1, \dots, y_m]$. Before stating the main theorem we need some preparation.

Lemma 2.1. *Let M be a finitely generated bigraded A -module. Then the following holds:*

- (a) *There exists an integer s such that $M_{k+1} = \mathfrak{n}M_k$ for $k \geq s$.*
- (b) *The Krull dimension $\dim M_k$ of M_k is constant for all $k \gg 0$.*

We set $\text{ldim } M = \lim_{k \rightarrow \infty} \dim M_k$.

- (c) *Let $M' = \bigoplus_{k \geq k_0} M_k$ where k_0 is chosen such that $\dim M_k = \text{ldim } M$ and $M_{k+1} = \mathfrak{n}M_k$ for all $k \geq k_0$. Then*
 - (i) $\dim M'/\mathfrak{n}M' = \text{ldim } M' = \text{ldim } M$;
 - (ii) $\dim M'/\mathfrak{m}M' = \dim M/\mathfrak{m}M$.

Proof. (a) Set $N = M/\mathfrak{n}M$. Then N is a finitely generated bigraded A -module with $\mathfrak{n}N = 0$. Let n_1, \dots, n_r be a set of bihomogeneous generators of N with $\deg n_i = (l_i, k_i)$, and let s be the maximum of the k_i . We claim that $N_k = 0$ for $k > s$. Indeed, let $u \in N_k$. We may assume that u is bihomogeneous, say $\deg u = (j, k)$. Then there exist bihomogeneous elements f_1, \dots, f_r with $\deg f_i = (a_i, b_i)$ and $u = \sum_{i=1}^r f_i n_i$ such that $b_i + k_i = k$. It follows that $b_i > 0$ for all i . Therefore each monomial in the support of f_i contains as a factor a monomial in the y_j of degree b_i . Since all y_j annihilate each n_i , we see that $u = 0$, and hence $N_k = 0$. Consequently, $M_{k+1} = \mathfrak{n}M_k$ for $k \geq s$.

(b) Let s be as in (a). Then $M_{k+1} = \mathfrak{n}M_k$ for all $k \geq s$. So $\text{Ann}_S M_k \subseteq \text{Ann}_S M_{k+1}$ for all $k \geq s$. Since S is Noetherian, there exists $k_0 \geq s$ such that $\text{Ann } M_k = \text{Ann } M_{k+1}$ for all $k \geq k_0$. Then $\dim M_{k+1} = \dim M_k$ for all $k \geq k_0$.

(c)(i) Since $M'/\mathfrak{n}M' \cong M_{k_0}$ it follows that $\dim M'/\mathfrak{n}M' = \dim M_{k_0} = \text{ldim } M$.

(c)(ii) $M'/\mathfrak{m}M' = \bigoplus_{k \geq k_0} M_k/\mathfrak{m}M_k$ and $M/\mathfrak{m}M = \bigoplus_k M_k/\mathfrak{m}M_k$. Therefore, $M'/\mathfrak{m}M'$ is an S' -submodule of $M/\mathfrak{m}M$ and

$$(M/\mathfrak{m}M)/(M'/\mathfrak{m}M') = \bigoplus_{k < k_0} M_k/\mathfrak{m}M_k.$$

Since there are only finitely many $k < k_0$ with $M_k \neq 0$, it follows that

$$\dim(M/\mathfrak{m}M)/(M'/\mathfrak{m}M') = 0.$$

This implies that $\dim M/\mathfrak{m}M = \dim M'/\mathfrak{m}M'$, as desired. \square

In the following we use the convention that the zero polynomial has degree -1 .

Theorem 2.2. *Let M be a finitely generated bigraded A -module. Then for $k \gg 0$, $e_j^i(M_k)$ is a polynomial in k , and*

$$\deg e_j^i(M_k) \leq m + j - 1 \quad \text{for } j = 0, \dots, \text{ldim } M + i - 1,$$

and $e_j^i(M_k) = 0$ for $j > \text{ldim } M + i - 1$.

Proof. Let M' be defined as in Lemma 2.1. Since $M'_k = M_k$ for $k \gg 0$, we have that $e_j^i(M'_k) = e_j^i(M_k)$ for $k \gg 0$. Therefore, since $\dim M'/\mathfrak{m}M' = \dim M/\mathfrak{m}M$, we may replace M by M' , and hence may assume from the very beginning that M itself satisfies condition (c)(i) and (c)(ii) of Lemma 2.1.

Let $J = \text{Ann}_S(M/\mathfrak{n}M)$. Then $M/\mathfrak{n}M$ is a finitely generated module over the standard graded K -algebra $B = S/J$. We may assume that K is infinite, because otherwise we

may replace K by a suitable base field extension. By the graded Noether Normalization Theorem (see [1, Theorem 1.5.17]), there exist linear forms $z_1, \dots, z_d \in S$ such that B is a finitely generated $K[z_1, \dots, z_d]$ -module, where $d = \dim M/\mathfrak{n}M$. It follows that M is a finitely generated bigraded A' -module, where $A' = K[z_1, \dots, z_d, y_1, \dots, y_m] \subset A$ with $\deg z_i = (1, 0)$. Indeed, since $M/\mathfrak{n}M$ is a finitely generated bigraded B -module, and B is a finitely generated bigraded $A'/\mathfrak{n}A' = K[z_1, \dots, z_d]$ -module, it follows that $M/\mathfrak{n}M$ is a finitely generated bigraded $A'/\mathfrak{n}A'$ -module. Therefore, by Nakayama's Lemma, M is a finitely generated bigraded A' -module.

Now let \mathbb{F} be a bigraded minimal free A' -resolution of M with

$$F_r = \bigoplus_s A'(-a_{rs}, -b_{rs}) \quad \text{for all } r.$$

Then \mathbb{F}_k is a graded free $K[z_1, \dots, z_d]$ -resolution of M_k , where \mathbb{F}_k is the k th graded piece of \mathbb{F} which is obtained from \mathbb{F} by restricting the differentials of \mathbb{F} to the graded components $(F_r)_k = \bigoplus_j (F_r)_{(j,k)}$. It follows that $P_{M_k}^i(x) = \sum_r (-1)^{r+1} P_{(F_r)_k}^i(x)$. Since each $(F_r)_k$ is a free S -module, each $(F_r)_k$ has dimension n .

We write $\sum_r (-1)^{r+1} P_{(F_r)_k}^i(x) = \sum_j (-1)^j f_{j,k}^i \binom{x+d-j}{d-j}$. Since the coefficients $f_{j,k}^i$ are linear combination of terms of the form $e_j^i(A'(-a, -b)_k)$, it follows from Proposition 1.5 that the $f_{j,k}^i$ are polynomials in k of degree $\leq m + j - 1$.

Since by Lemma 2.1, $\dim M_k = d$ for all $k \gg 0$, we see that $e_j^i(M_k) = f_{j,k}^i$ for all i, j and $k \gg 0$. This yields the desired result. \square

Remark 2.3. The fact that $\deg e_j^i$ independent on i is also consequence of Corollary 1.2.

Example 2.4. Let $S = K[x_1, x_2]$, $\mathfrak{m} = (x_1, x_2)$ and $\mathcal{R}(\mathfrak{m}) = \bigoplus_{k \geq 0} \mathfrak{m}^k$. Let $A = K[x_1, x_2, y_1, y_2]$ with bigrading defined by $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (1, 1)$, for $i = 1, 2$. The natural map defined by $x_i \mapsto x_i$ and $y_i \mapsto x_i t$, for $i = 1, 2$, is then a surjective homomorphism. So $\mathcal{R}(\mathfrak{m})$ has a bigraded free resolution of the form

$$0 \rightarrow A(-2, -1) \rightarrow A \rightarrow \mathcal{R}(\mathfrak{m}) \rightarrow 0$$

Hence $e_j^i(\mathfrak{m}^k) = e_j^i(A_k) - e_j^i(A(-2, -1)_k)$. One has $e_j^i(A_k) = (k+1) \binom{k}{j}$ and $e_j^i(A(-2, -1)_k) = k \binom{k+1}{j}$. So $e_j^i(\mathfrak{m}^k) = \frac{k(k-1) \cdots (k-j+2)}{j} (1-j)(k+1)$. Therefore $\deg(e_j^i(\mathfrak{m}^k)) = j$, and by Theorem 2.2 our upper bound is $j + 1$.

In the special case that all p_i are the same, we can improve the upper bound for the degree of the higher iterated Hilbert coefficients as follows:

Theorem 2.5. Assume that $p_1 = p_2 = \dots = p_m = p$, and let M be a finitely generated bigraded A -module. Then for $k \gg 0$, $e_j^i(M_k)$ is a polynomial in k , and

$$\deg e_j^i(M_k) \leq \dim M/\mathfrak{m}M + j - 1 \quad \text{for } j = 0, \dots, \text{l dim } M + i - 1,$$

and $e_j^i(M_k) = 0$ for $j > \text{l dim } M + i - 1$.

Proof. By using the Noether Normalization Theorem, we may replace, as in the proof of Theorem 2.2, A by $A'' = K[z_1, \dots, z_d, w_1, \dots, w_{d'}]$ where $d = \text{l dim } M$ and $d' = \dim M/\mathfrak{m}M$. Then by computing the higher iterated Hilbert polynomial by using a bigraded free A'' -resolution of M , yields, as in the proof of Theorem 2.2, the desired conclusion. \square

The given upper bound for the degree of the higher iterated Hilbert coefficients of a bigraded A -module as given in Theorem 2.5 is in general sharp, for example for $M = A$. In more special cases it may not be sharp. Indeed, let $I \subset S$ be a graded ideal generated by m homogeneous polynomials of degree p , and let $\mathcal{R}(I) = \bigoplus_{k \geq 0} I^k$ the Rees ring of I . Then $\mathcal{R}(I)$ is a bigraded A -algebra with $\mathcal{R}(I)_k \cong I^k$ and $\dim \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = \ell(I)$, which by definition is the analytic spread of I . Thus we have

Corollary 2.6. *Let $I \subset S$ be a graded ideal generated in a single degree. Then for all $k \gg 0$, $e_j^i(I^k)$ is a polynomial function of degree $\leq \ell(I) + j - 1$.*

In case that I is \mathfrak{m} -primary, one has $e_0^i(I^k) = 1$ for all i and k so that $\deg e_0^i(I^k) = 0$, while the formula in Corollary 2.6 gives the degree bound $n - 1$, since $\ell(I) = n$.

3. THE HIGHER ITERATED HILBERT COEFFICIENTS OF THE GRADED COMPONENTS OF TOR AND EXT

Let M be a graded S -module and $N = \bigoplus_{i,j \in \mathbb{N}} N_{(i,j)}$ bigraded A -module. We will see that $\mathrm{Tor}_i^S(M, N)$ and $\mathrm{Ext}_S^i(M, N)$ are naturally bigraded A -modules. Thus we may then study the higher iterated Hilbert coefficients of the graded components of these modules.

Let U be a finitely generated graded S -module, and V be a finitely generated bigraded A -module. We first notice that

$$U \otimes_S V \quad \text{and} \quad \mathrm{Hom}_S(U, V)$$

are bigraded A -modules. Indeed,

$$(U \otimes_S V)_{(c,d)} = \bigoplus_k U_k \otimes_K V_{(c-k,d)},$$

and

$$\mathrm{Hom}_S(U, V)_{(c,d)} = \{f \in \mathrm{Hom}_S(U, V) : f(U_i) \subset V_{(i+c,d)} \text{ for all } i\}.$$

With this bigraded structure as described above we have

$$(U \otimes_S V)_k = U \otimes_S V_k \quad \text{and} \quad \mathrm{Hom}_S(U, V)_k = \mathrm{Hom}_S(U, V_k) \quad \text{for all } k.$$

Lemma 3.1. *Let M be a finitely generated graded S -module and N finitely generated bigraded A -module. Then, for all i , $\mathrm{Tor}_i^S(M, N)$ and $\mathrm{Ext}_S^i(M, N)$ are finitely generated bigraded A -modules, and*

$$\mathrm{Tor}_i^S(M, N)_k \cong \mathrm{Tor}_i^S(M, N_k) \quad \text{and} \quad \mathrm{Ext}_S^i(M, N)_k \cong \mathrm{Ext}_S^i(M, N_k) \quad \text{for all } i \text{ and } k.$$

Proof. Let \mathbb{F} bigraded free A -resolution of N . Then

$$\mathrm{Tor}_i^S(M, N)_k = H_i(M \otimes_S \mathbb{F})_k = H_i((M \otimes_S \mathbb{F})_k) = H_i(M \otimes_S \mathbb{F}_k) = \mathrm{Tor}_i^S(M, N_k).$$

Here we used that taking the graded components can be exchanged with taking homology, and we also used that \mathbb{F}_k is a graded free S -resolution of N_k .

In order to compute $\mathrm{Ext}_S^i(M, N)$ we choose a graded free S -resolution \mathbb{F} for M . Then $\mathrm{Hom}_S(\mathbb{F}, N)$ is a complex of bigraded A -modules, and $\mathrm{Ext}_S^i(M, N) = H^i(\mathrm{Hom}_S(\mathbb{F}, N))$ has a natural bigraded structure. Moreover,

$$\begin{aligned} \mathrm{Ext}_S^i(M, N)_k &= H^i(\mathrm{Hom}_S(\mathbb{F}, N))_k = H^i(\mathrm{Hom}_S(\mathbb{F}, N)_k) = H^i(\mathrm{Hom}_S(\mathbb{F}, N_k)) \\ &= \mathrm{Ext}_S^i(M, N_k) \quad \text{for all } k. \end{aligned}$$

□

As a consequence of Lemma 2.1 we obtain

Corollary 3.2. *Let M be a finitely generated graded S -module and N finitely generated bigraded A -module. Then the Krull dimension of the finitely generated graded S -modules $\mathrm{Tor}_i^S(M, N)_k$ and $\mathrm{Ext}_S^i(M, N)_k$ are constant for $k \gg 0$.*

Next we want to study further the graded S -modules $\mathrm{Tor}_i^S(M, N)_k$ and $\mathrm{Ext}_S^i(M, N)_k$. By the preceding corollary, their Hilbert polynomials have constant degree for large k . For $\mathrm{Tor}_i^S(M, N)_k$, these degrees can be bounded as follows

Proposition 3.3. *With the notation and assumptions as before, we have*

$$\dim \mathrm{Tor}_{i+1}^S(M, N)_k \leq \dim \mathrm{Tor}_i^S(M, N)_k \quad \text{for all } k.$$

In particular, $\dim \mathrm{Tor}_{i+1}^S(M, N)_k \leq \dim(M \otimes_S N_k)$ for all k , and hence for $k \gg 0$, the degree of the j th iterated Hilbert polynomial of $\mathrm{Tor}_i^S(M, N)_k$ is less than or equal to $\dim(M \otimes_S \mathrm{ldim} N) + j - 1$.

Proof. Let $T = K[y_1, \dots, y_n]$. We may view N_k a graded T -module by setting $y_i u := x_i u$ for all $i = 1, \dots, n$ and $u \in N_k$. So $M \otimes_K N_k$ has the natural structure of an $S \otimes_K T$ -module, and

$$\mathrm{Tor}_i^S(M, N)_k \cong \mathrm{Tor}_i^S(M, N_k) \cong H_i(x_1 - y_1, \dots, x_n - y_n; M \otimes_K N_k),$$

where $H_i(_)$ denotes Koszul homology. (see [2, Chapter IX, Theorem 2.8] and [10, page 101])

Thus in order to see that $\dim \mathrm{Tor}_{i+1}^S(M, N)_k \leq \dim \mathrm{Tor}_i^S(M, N)_k$ it suffices to show that whenever W is a graded module over a polynomial ring R , and \mathbf{x} is a finite sequence of elements of R , then $\dim H_{i+1}(\mathbf{x}; R) \leq \dim H_i(\mathbf{x}; W)$ for all i . To see this, let P be in the support of $H_{i+1}(\mathbf{x}; W)$. Then we have to show that P is in the support of $H_i(\mathbf{x}; W)$. Since $H_{i+1}(\mathbf{x}; W_P) = H_{i+1}(\mathbf{x}; W)_P \neq 0$ it follows from [1, Exercise 1.6.31] that $H_i(\mathbf{x}; W_P) = H_i(\mathbf{x}; W)_P \neq 0$, and the desired conclusion follows. \square

Corollary 3.4. *Let M be a finitely generated graded S -module and N finitely generated bigraded A -module. Then for all $k \gg 0$, $e_j^i(\mathrm{Tor}_l^S(M, N_k))$ and $e_j^i(\mathrm{Ext}_S^l(M, N_k))$ are polynomials in k of degree at most $m - 1 + j$*

In special the case that $p_i = p$ for all i , the degree of $e_j^i(\mathrm{Tor}_l^S(M, N_k))$ is bounded by $\dim \mathrm{Tor}_l^S(M, N)/\mathfrak{m} \mathrm{Tor}_l^S(M, N) + j - 1$ and the degree of $e_j^i(\mathrm{Ext}_S^l(M, N_k))$ is bounded by $\dim \mathrm{Ext}_S^l(M, N)/\mathfrak{m} \mathrm{Ext}_S^l(M, N)$.

Corollary 3.5. *Let M be a graded S -module, and $I \subset S$ a graded ideal. Then for $l > 1$, $e_j^i(\mathrm{Tor}_l^S(M, S/I^k))$ is polynomial function in k of degree less than or equal to $v(I) + j - 1$ where $v(I)$ denotes the number of generators of I . If all generators of I have the same degree then $v(I)$ can be replaced by $\dim \mathcal{R}(I)/\mathrm{Ann}_S(M)\mathcal{R}(I)$.*

Proof. The exact sequence

$$0 \rightarrow I^k \rightarrow S \rightarrow S/I^k \rightarrow 0,$$

implies that $\mathrm{Tor}_l^S(M, S/I^k) \cong \mathrm{Tor}_{l-1}^S(M, I^k) \cong \mathrm{Tor}_{l-1}^S(M, \mathcal{R}(I))_k$ for all $l > 1$ where $\mathcal{R}(I)$ is the Rees ring of I . So, by Corollary 3.4, we see that $e_j^i(\mathrm{Tor}_l^S(M, S/I^k))$ is a polynomial in k for all $k \gg 0$ of degree less than or equal to $v(I) + j - 1$. In the special case that all generators of I have the same degree, Corollary 3.4 implies that

$$\begin{aligned} \deg(e_j^i(\mathrm{Tor}_l^S(M, S/I^k))) &\leq \dim \mathrm{Tor}_{l-1}^S(M, I^k) + j - 1 \leq \dim(M \otimes_S \mathcal{R}(I)) + j - 1 \\ &\leq \dim(S/\mathrm{Ann}_S(M) \otimes_S \mathcal{R}(I)) + j - 1 \\ &\leq \dim \mathcal{R}(I)/(\mathrm{Ann}_S(M)\mathcal{R}(I)) + j - 1. \end{aligned}$$

\square

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